Limitation of the Q factor due to the thermo elastic damping in flexural Langasite and Langatate resonators.

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***Abstract*—We have measured the heat capacity and the thermal conductivity of langasite and langatate crystals. These data are used to compute the limit Q factor due to the thermoelastic damping of resonators vibrating in flexure mode.**

I. Introduction

In triclinic crystals thermoelastic damping of vibration arises when a local deformation, including shear strain, exists. In isotropic materials, this effect has been first taken into account by Zener [1] to explain the quality factor of steel beam vibrating in flexure mode. It also exists for plane waves if the mode is not a pure shear mode. Langasite and crystals of the family are materials with an acoustic damping lower than quartz. They have an interest for high frequency resonators because the Q factor we can expect, and for low frequency devices of the MEMS family because their high piezoelectric coefficients. The prediction of the limitation of the Q factor due to the thermoelastic damping is governed by the thermal conductivity and heat capacity.

For the first time we give the measured values for these constants. Langasite (LGS) and Langatate (LGT) crystals are of 32 cristallographic class. So there are only two independent thermal conductivity coefficients along X (or Y) and Z axis.

These data are then employed to predict the limit Q factor of resonators vibrating in flexure mode. We have limited our investigations to resonators of rectangular cross section and we recall the theory [2], [3] of thermoelastic damping. As a starting point we use the constitutive relations commonly used in piezoelectricity.

II. Thermal properties of LGS and LGT

A. Heat capacity

The heat capacity of LGS and LGT have been measured at various temperature by using a calorimeter made in copper. In order to reduce the thermal exchanges it was put in a thermally regulated shield under high vacuum (pressure less than $1\cdot10^{-5}$ Torr). A linear correction was made to compensate the thermal exchange due to the thermal radiation.

We have found that in the range of 20°C to 150°C the specific thermal capacity at constant stress are:

LGS:

$C^T = 405\cdot[1+4.1\cdot10^{-3}(T-20)]$ J/kg/°C

LGT:

$C^T = 389\cdot[1+3.4\cdot10^{-3}(T-20)]$ J/kg/°C

B. Thermal conductivity

Thermal conductivities have been obtained directly by placing a bar between two blocks of copper, the lower of which is maintained in contact with a temperature regulated oven and an electric power applied to the upper block and by measuring the variation of difference of temperature between the two blocks when the electric power was applied or not . To avoid thermal exchanges between the sample and the upper block, we have operated under high vacuum and a regulated thermal shield was put around the upper block.

The block sizes are 16 x 16 x 80 mm (length along Y axis)

Obtained data are:

LGS:

$\gamma_1 = 1.39\cdot[1+1.3\cdot10^{-3}(T-20)]$ J/sm°C

$\gamma_3 = 1.81\cdot[1+0.92\cdot10^{-3}(T-20)]$ J/sm°C

LGT:

$\gamma_1 = 1.40\cdot[1+1.3\cdot10^{-3}(T-20)]$ J/sm°C

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$\gamma_3 = 1.86\cdot [1+0.71\cdot 10^{-3}(T-20)]$ J/sm°C

III. THERMOELASTIC DAMPING OF FLEXURAL RESONATORS

The strains and the variation of entropy σ are given by the constitutive relations [4]

$$S_i = \sum_j s^{\theta}{}_{ij} T_j \quad + \quad \alpha_i \theta$$
$$\sigma = \sum_i \alpha_i T_i \quad + \quad \frac{C^T \theta}{\rho\, T_0} \tag{1}$$

where :

$s^{\theta}{}_{ij}$: isothermal elastic stiffness

α_i : thermal expansion coefficients

θ = T-To: variation of absolute temperature from reference temperature To

C^T: heat capacity per unit mass at constant stress

ρ : mass per unit volume.

We consider a beam of length L along x_2 and dimensions h_1 and h_3 along x_1 and x_3 respectively. If the flexure (displacement u) occurs in the x_i direction (i = 1 or 3), the strain in the x_2 direction is

$$S_2 = -x_i \frac{\partial^2 u_i}{\partial x_2^2} \tag{2}$$

and the stress is

$$T_2 = -\frac{x_i}{s_{22}} \frac{\partial^2 u_i}{\partial x_2^2} - \frac{\alpha_2}{s_{22}} \theta \tag{3}$$

Putting this expression in the equation

$$\rho A \frac{\partial^2 \mathrm{u}}{\partial \mathrm{t}^2} - \frac{\partial^2 \mathrm{M}}{\partial \mathrm{x}_2^2} = 0 \tag{4}$$

governing the flexure of the beam (A=area of cross section, M=bending moment), the equation of motion is

$$\rho A \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial^2}{\partial x_2{}^2}\left[\frac{I_i}{s_{22}} \frac{\partial^2 u_i}{\partial x_2{}^2} + \frac{\alpha_2}{s_{22}} I_{Ti}\right] = 0 \tag{5}$$

where

$$I_i = \int_{-h_1/2}^{+h_1/2} \int_{-h_3/2}^{+h_3/2} x_i^2 dx_1 dx_3 = \frac{h_1 h_3 h_i^2}{12} \tag{6}$$

is the moment of the beam and

$$I_{Ti} = \int_{-h_1/2}^{+h_1/2} \int_{-h_3/2}^{+h_3/2} x_i\, \theta\, dx_1 dx_3 \tag{7}$$

is the contribution due to non uniformity of the temperature. From (1.1) the rate of increase of heat is:

$$\frac{dQ}{dt} = \gamma_i \frac{\partial^2 \theta}{\partial x_i^2} = \frac{\partial}{\partial t}\left(\alpha_2 S_2 T + \rho\, C^T \theta\right) \tag{8}$$

If γ_i is the heat conduction in the x_i direction, the heat law is expressed as

$$\rho C \frac{\partial \theta}{\partial t} = \gamma_i \frac{\partial^2 \theta}{\partial x_i^2} + x_i T_0 \frac{\partial}{{}_0 \partial t} \frac{\partial^2 u_i}{\partial x_2^2} \tag{9}$$

where

$$C = C^T - \frac{\alpha_2^2}{s_{22}} T_0 \tag{10}$$

As solution of (5) and (9) we take

$$\theta = \theta_0 e^{j\omega t}; \quad u_i = u_0\, e^{j\omega t} \tag{11}$$

From (9), θ_0 is solution of

$$\chi_i \frac{\partial^2 \theta_0}{\partial x_i^2} = j\omega \left[\theta_0 - \frac{\alpha_2\, x_i\, T_0}{s_{22} \rho\, C} \frac{\partial^2 u_0}{\partial x_2^2}\right] \tag{12}$$

where χ_i is the thermal diffusivity coefficient

$$\chi_i = \frac{\gamma_i}{\rho\, C} \tag{13}$$

We consider that there is no heat flux on the face of the beam. The boundaries conditions are

$$\frac{\partial \theta_0}{\partial x_i} = 0 \qquad at \qquad x_i = \pm \frac{h}{2} \tag{14}$$

Putting

$$k = (1-j)\sqrt{\frac{\chi_i \omega}{2}} \tag{15}$$

from (12), the θ_0 solution is:

$$\theta_0 = \frac{\alpha_2\, T_0}{s_{22}\, C} \frac{\partial^2 u_0}{\partial x_2^2}\left[x_i - \frac{\sin kx_i}{k \cos \frac{kh_i}{2}}\right] \tag{16}$$

The extra moment due to the temperature variation (7) becomes

$$I_T = \frac{\alpha_2\, T_0}{s_{22}\, C} \frac{bh^3}{12}\left[1 + \frac{12}{k^2 h_i{}^2}\left(1 - \frac{2}{kh_i} tg \frac{kh_i}{2}\right)\right] \frac{\partial^2 u_0}{\partial x_2^2} \tag{17}$$

Putting this expression in (5) one gets

$$\omega^2 u_0 = \frac{E_\omega I}{\rho A} \frac{\partial^4 u_0}{\partial x_2^4} \tag{18}$$

where the Young modulus E_ω is now a function of the frequency

$$E_\omega = \frac{1}{s_{22}}\left[1 + \frac{\alpha_2^2\, T_0}{s_{22}\, C}\left(1 + f(\omega)\right)\right]$$
$$f(\omega) = \frac{24}{k^3 h_i{}^3}\left(\frac{kh_i}{2} - tg \frac{kh_i}{2}\right) \tag{19}$$

From (18) the eigenfrequency is computed by the usual way.

$$\omega_n = \sqrt{\frac{E_\omega I}{\rho\ A}\frac{a_n^2}{L^2}} \cong \omega_{0n}\left[1+\frac{\alpha_2^2 T_0}{2 s_{22}\ C}\left(1+f(\omega)\right)\right] \quad (20)$$

$$\omega_{0n} = \sqrt{\frac{I}{\rho\ A\, s_{22}}\frac{a_n^2}{L^2}}$$

a_n is a parameter which depends of the mechanical boundaries conditions and of the rank of mode (a_1=1.875, a_2=4.694, a_3=7.855… for a clamped-free beam).

Because de k number in (15) is complex, the eigenfrequency is so one, which means that there is a damping of the vibration.

The Q factor can be defined as

$$Q = \frac{1}{2}\frac{\mathrm{Re}(\omega)}{\mathrm{Im}(\omega)} \cong \frac{1}{2}\frac{\omega_{0n}}{\mathrm{Im}(\omega)} \quad (21)$$

Finally the Q factor of the resonator is given by [2],[3]

$$Q_t^{-1} = \frac{\alpha_2^2 T_0}{s_{22}\ \rho\ C}\left(\frac{6}{\xi_i^{\ 2}} - \frac{6}{\xi^3}\frac{\sin\ \xi_i + \sinh\ \xi_i}{\cos\ \xi_i + \cosh\ \xi_i}\right) \quad (22)$$

$$\xi_i = h\sqrt{\frac{\omega_{on}}{2\chi_i}} \quad (23)$$

The Q factor exhibits a minimum value for ξ = 2.225, which corresponds to the frequency

$$f_{0\min} = 1.576\ \chi_i\ h_i^{-2} \quad (24)$$

This value is

$$Q_{\min} = \frac{s_{22}\ \rho\ C}{0.494\ \alpha_2^2\ T_0} \quad (25)$$

At high frequency the Q factor (22) becomes proportional to the frequency

$$Q \approx \frac{s_{22}}{12}\left(\frac{\rho\, C}{\alpha_2}\right)^2 \frac{h_i^2\ \omega}{\gamma_i T_0} = 0.825\frac{Q_{\min}}{f_{0\min}} f \quad (26)$$

IV. Cases of LGS and LGT

In a first time, we consider a resonator of square section of width h, the length being along the Y cristallographic axis and the main faces perpendicular to X and Z axis.

The figure 1 gives the variation of the minimum Q factor for both crystals as well as for quartz. The used values for the first and second order thermal expansion coefficients of LGS and LGT are those from [5]. Thermal dependence of s_{22} is also taken into account, coefficients are those given in [6]. For quartz, the corresponding data are from [7] and [8].

The specific heat capacity is obtained from [9]:

$C = 694 + 1.91 \cdot T - 3.38 \cdot 10^{-3} \cdot T^2$ J / kg / °C.

Because their lower thermal expansions, both crystals exhibit higher values than quartz.

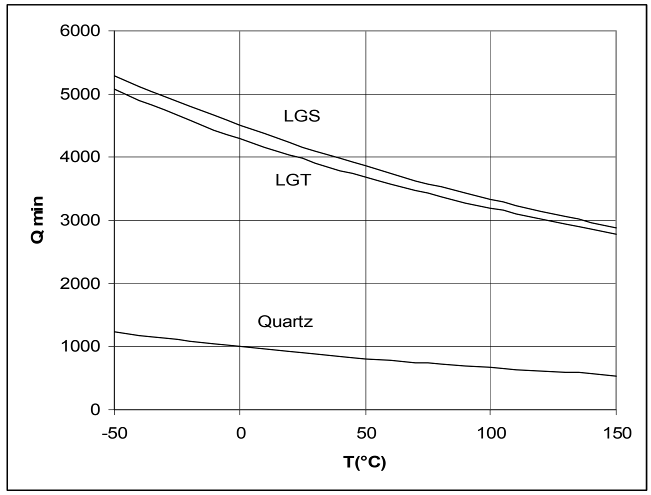

Figure 1 Variations of minimum Q factor Q_{min} with respect to temperature.

In practice resonators operate at higher frequencies than f_{omin}. The lower the minimum frequency f_{omin}, the higher the Q factor.

Variations of products f_{omin} h² are given on figures 2 and 3 for vibration in the X and Z directions. For quartz we have used the values (T in °C)

$\gamma_1 = 6.94 - 0.024 \cdot T + 1.52 \cdot 10^{-4} \cdot T^2 - 6.63 \cdot 10^{-7} \cdot T^3$ J/sm°C

$\gamma_3 = 13.18 - 0.065 \cdot T + 2.41 \cdot 10^{-4}\ T^2 - 6.9 \cdot 1 \cdot 10^{-8} \cdot T^3$ J/sm°C

obtained from [9] and [10].

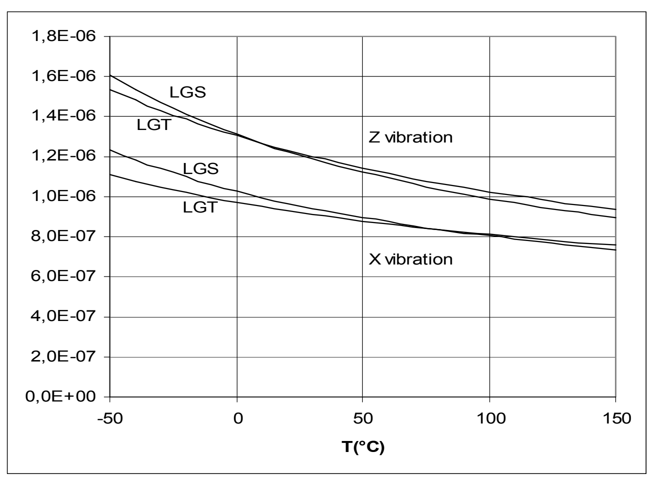

Figure 2 Variations of frequency of minimal Q value constant f_{omin} h² with respect to temperature for LGS and LGT.

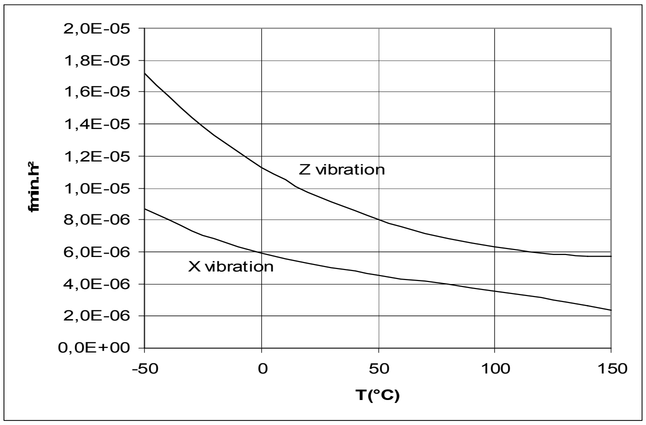

Figure 3 Variations of frequency of minimal Q value f_{omin} h² with respect to temperature for quartz.

As an example the table I give the Q factor one can expect for resonators of square section 400 x 400 micron having a resonant frequency of 10 kHz if the Q factor is only limited by the thermoelastic damping. In this example $f >> f_{omin}$ and the approximation (26) is valid. Q_{min} and f_{omin} of all metarials have similar thermal behaviors and the Q factors don't exhibit a strong variation with the temperature.

For comparison, by using the data given in [11], we obtain a Q value of 51600 for the same resonator made in silicon at 25°C.

TABLE I THERMOELASTIC Q FACTOR (in 10^6) FOR RESONATORS VIBRATING AT 10 KHZ

	X Vibration			Z Vibration		
Temperature	LGS	LGT	Quartz	LGS	LGT	Quartz
-40 °C	3.30	3.31	0.112	2.47	2.41	0.058
+ 25 °C	3.25	3.09	0.125	2.51	2.33	0.069
+ 120° C	2.96	2.70	0.132	2.35	2.14	0.078

The thermoelastic damping is also function of the orientation of resonators. Figure 4 and 5 show the variations of Q_{min} and f_{omin} in the case of a singly rotated resonator X+θ. (θ = rotation about X axis), when the vibration occurs in the X direction.

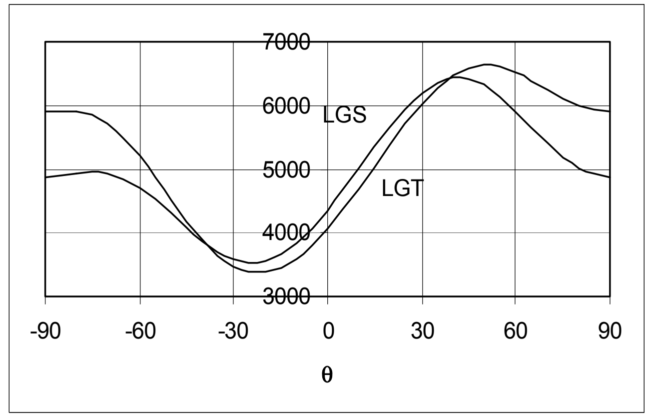

Figure 4. Variations of minimal Q value Q_{min} as a function of cut angle (X+θ cut).

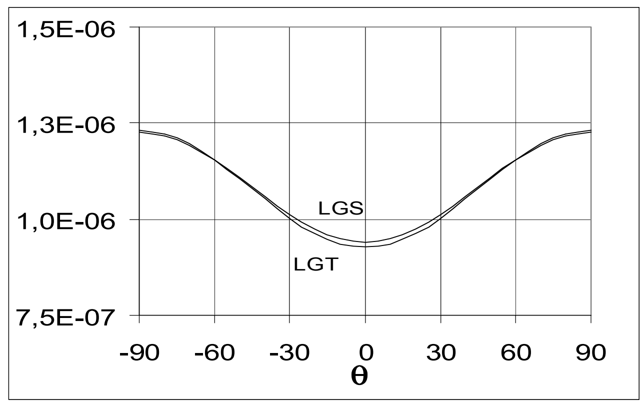

Figure 5 Variations of frequency of minimal Q value constant f_{omin} h² as a function of cut angle (X+θ cut).

V. CONCLUSION

LGS and LGT have low thermal expansion coefficients and low thermal conductivity in comparison to materials like quartz or silicon. So, they present a lower thermoelastic damping.

It can be expected that new classes of high performances vibrating beam sensors, able to operate at high temperatures, can be designed in the near future.

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